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Deformation space of discontinuous groups $\mathbb{Z}^k$ for a nilmanifold $\mathbb{R}^{k+1}$

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Abstract

This article is a brief summary of the lecture delivered at the RIMS workshop, July 2005. We consider the deformation of a discontinuous group acting on the Euclidean space by affine transformations. A distinguished phenomenon here is that even a 'small' deformation as discrete subgroups may not preserve the condition of properly discontinuous actions. In order to understand the local structure of the deformation space of discontinuous groups, we introduce the concept 'stability' and 'local rigidity' of discontinuous groups for homogeneous spaces. As a test case, we provide a concrete and explicit description of the deformation space of $\mathbb{Z}^k$ acting properly discontinuously on $\mathbb{R}^{k+1}$ by affine nilpotent transformations. This is carried out by characterizing the set of properly discontinuous groups in the deformation space of discrete subgroups.

1 Introduction

Our concern of this article is with the deformation of discontinuous groups acting on a non-Riemannian homogeneous space. Here, by a **discontinuous group**, we mean a discrete group acting properly discontinuously on a topological space.

A distinguished phenomenon in the non-Riemannian setting is that a deformation of discrete subgroups may destroy the condition of properly discontinuous actions. Therefore, it is crucial to tell whether a deformed action is properly discontinuous or not.

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Deformation as discontinuous groups

$\parallel$

Deformation of discrete subgroups
(group theory)
$+$
Properly discontinuous actions
(action)

As in the above box, **deformation as discontinuous groups** consists of two ingredients. One is to deform discrete subgroups, and the other is to keep the action to be properly discontinuous. The former is the study of group structure, and the latter is the study of group actions.

1.1 Riemannian case

We start with a very special case. Suppose a Lie group G acts on a **Riemannian manifold** X by isometry. Then, any subgroup Γ of G also acts on X by isometry. In this setting, it turns out that the following two statements are equivalent.

(group theory)

Γ : discrete subgroup of G

$\Updownarrow$ X: Riemannian

(action)

$\Gamma \curvearrowright X$ properly discontinuous

Example 1.1.1. (Riemannian case)

$G = \mathrm{PSL}(2,\mathbb{R}) \overset{\text{isometry}}{\curvearrowright} X = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$ (Poincaré upper half plane),

$\cup$

$\Gamma = \pi_1(M_g)$, M_g is a closed Riemann surface with genus $g \geq 2$.

$\Longrightarrow$ Deformation of Γ in G
$\vdots$
Teichmüller space of M_g

Remark 1.1.2. The above equivalence does no longer hold if X is a **pseudo-Riemannian manifold** acted isometrically by G.

1.2 Non-Riemannian case

As we mentioned, deformation of discontinuous groups in the Riemannian case is just equivalent to deformation of discrete subgroups. However, our interest here is in the deformation of discontinuous groups in a more general setting, namely, in the non-Riemannian case. Then, we note:

$$\begin{array}{ll} G \curvearrowright X & : \text{non-Riemannian} \\ \cup & \\ \Gamma & : \text{subgroup} \end{array} \qquad \boxed{\begin{array}{c} \Gamma : \text{discrete subgroup} \\ \text{(group theory)} \\ \Uparrow \quad \not\Downarrow \\ \Gamma \curvearrowright X : \text{properly discontinuous.} \\ \text{(action)} \end{array}}$$

In the above setting, if the action is properly discontinuous, then the group Γ is automatically discrete in G. However, the converse is not necessarily true, that is, the isometric action of a discrete subgroup on a pseudo-Riemannian manifold (X, g) is not always properly discontinuous unless the signature g is definite.

The most typical example to illustrate this phenomenon is so-called **Calabi-Markus phenomenon** which was first observed in the Lorentz manifold ([CM62]). In fact, even though the isometry group G contains a rich family of discrete subgroup, it can happen that there is essentially no discontinuous group. In general, by **Calabi-Markus phenomenon** for a homogeneous space G/H, we shall mean that G/H admits only finite discontinous groups.

For semisimple symmetric spaces G/H, **Calabi-Markus phenomenon** occurs if and only if the rank condition "$\mathbb{R}\text{-}\operatorname{rank} G = \mathbb{R}\text{-}\operatorname{rank} H$" is satisfied ([K89]). Thus, discrete subgroups and discontinuous groups can be totally different in the non-Riemannian setting.

2 Formulation

2.1 Deformation space

Here, let us formulate "deformation" of discontinuous groups.

Suppose Γ is a finitely generated discrete group, and G is a Lie group. First, we denote by $\operatorname{Hom}(\Gamma, G)$, the set of all **group homomorphisms** from Γ into G. That is:

$$\boxed{\operatorname{Hom}(\Gamma, G) := \{\varphi : \Gamma \to G \ : \ \varphi \text{ is a group homomorphism}\}}$$

We topologize the set $\operatorname{Hom}(\Gamma, G)$ by pointwise convergence.

Let X be a Hausdorff, locally compact topological space, and G acts continuously on X. Next, we define its subset $R(\Gamma, G; X)$ as follows:

$$\boxed{R(\Gamma, G; X) := \left\{ \varphi \in \operatorname{Hom}(\Gamma, G) : \begin{array}{l} \bullet\ \varphi \text{ injective} \\ \bullet\ \varphi(\Gamma) \curvearrowright X \text{ properly discont. \& free} \end{array} \right\}}$$

Then, for each such φ, the quotient space $\varphi(\Gamma)\backslash X$ becomes a Hausdorff topological space, on which a manifold structure is canonically defined so that the natural quotient map

$$X \to \varphi(\Gamma)\backslash X$$

is a local homeomorphism. Then, the quotient space $\varphi(\Gamma)\backslash X$ enjoys locally the same geometric structure with X. The quotient space $\varphi(\Gamma)\backslash X$ is called a **Clifford-Klein form** of X.

Thus, we may interpret $R(\Gamma, G; X)$ as the parameter space of Clifford-Klein forms $\varphi(\Gamma)\backslash X$ with parameter φ.

To be more precise about the parameter φ of Clifford-Klein forms $\varphi(\Gamma)\backslash X$, we have to take 'unessential' deformation into consideration arising from inner automorphisms of G. We introduce the equivalence relation among $R(\Gamma, G; X)$ as follows:

Definition.
For $\varphi_1, \varphi_2 \in R(\Gamma, G; X)$,

$$\varphi_1 \sim \varphi_2 \Leftrightarrow \exists g \in G \text{ s.t. } \varphi_2 = g \circ \varphi_1 \circ g^{-1}$$

If $\varphi_1 \sim \varphi_2$, then we have naturally a diffeomorphism between two Clifford-Klein forms:

$$\begin{array}{ccc} \varphi_1(\Gamma)\backslash X & \underset{\text{homeo.}}{\xrightarrow{\sim}} & \varphi_2(\Gamma)\backslash X \\ \\ \varphi_1(\Gamma)x & \mapsto & \varphi_2(\Gamma)gx \end{array}$$

We say the set $\mathcal{T}(\Gamma, G; X)$ of equivalence classes is the **deformation space** of discontinuous groups for X:

Definition (see [K01])
$\mathcal{T}(\Gamma, G; X) := R(\Gamma, G; X)/G.$

In the case of a Riemannian symmetric space $X = G/K$, our terminology here is consistent with the usual one because any discrete subgroup Γ of G acts properly discontinuously on X.

In summary, we have the following sets and natural maps:

$$\begin{array}{ccc} R(\Gamma, G; X) & \subset & \mathrm{Hom}(\Gamma, G) \\ \downarrow & & \\ R(\Gamma, G; X)/G & =: & \underset{\text{Deformation space}}{\mathcal{T}(\Gamma, G; X)} \end{array}$$

Here is a prototype of the deformation space (in the Riemannian case).

Example.

$$\begin{cases} X = \{z \in \mathbb{C} : \mathrm{Im} z > 0\} \\ G = \mathrm{PSL}(2, \mathbb{R}) \\ \Gamma = \pi_1(M_g) \end{cases}$$

$\Rightarrow$

$\mathcal{T}(\Gamma, G; X) \simeq$ Teichmüller space of M_g

2.2 Rigidity and Stability

In general, the set of discontinuous groups $R(\Gamma, G; X)$ is not a manifold. There may be singularities in $R(\Gamma, G; X)$.

In order to understand the local structure of the deformation space, we now introduce two notions "Rigidity" and "Stability" for each element $\varphi_0 \in R(\Gamma, G; X)$.

We recall $R(\Gamma, G; X)$ is a subset of $\mathrm{Hom}(\Gamma, G)$, the set of all group homomorphisms from Γ into G. The group G acts on these two sets by inner automorphism.

We also recall that $R(\Gamma, G; X)$ is a topological space by pointwise convergence.

Definition ([K93, KN05]) Let $\varphi_0 \in R(\Gamma, G; X)$

Rigidity

$G \cdot \varphi_0$ is open in $\mathrm{Hom}(\Gamma, G)$

Stability

$R(\Gamma, G; X)$ contains a neighborhood of φ_0

In general, the following statements hold:

1) (Rigidity) $\Rightarrow$ (Stability).
2) $\dim \mathcal{T}(\Gamma, G; X) = 0 \Rightarrow$ any φ_0 is rigid

Definition.

$\varphi_0 : \Gamma \to G$ is **(locally) rigid** as a discontinuous group for X if (Rigidity) holds.
($\Leftrightarrow$ $[\varphi_0]$ is an isolated point in $\mathcal{T}(\Gamma, G; X)$.)

In the case of a Riemannian symmetric space, our terminology here is consistent with the traditional one that was introduced by Selberg and Weil [W64]. In this case, for any uniform lattice Γ in G, $R(\Gamma, G; X)$ becomes open in $\mathrm{Hom}(\Gamma, G)$, and in particular, any $\varphi_0 \in R(\Gamma, G; X)$ is stable in the above sense.

2.3 Rigidity theorems

Let G be a non-compact simple Lie group, and K its maximal compact subgroup. Now we give two examples of rigidity theorem. One is in the Riemannian case and the other is in the non-Riemannian case. We start with a Riemannian case.

Rigidity Theorem A (Selberg-Weil [W64])
Suppose $X = G/K$. Then,

$$\begin{aligned} &\exists\, \iota : \Gamma \hookrightarrow G \text{ cocompact} \\ &\text{s.t. } \iota \in R(\Gamma, G; X) \text{ is not rigid} \\ \Leftrightarrow\; &G \approx \mathrm{SL}(2, \mathbb{R}) \end{aligned}$$

Next, we consider a non-Riemannian case (group manifold case):

Rigidity Theorem B ([K98])
Suppose $X = (G \times G)/\operatorname{diag} G$. Then,

$$\begin{aligned} &\exists\, \iota : \Gamma \hookrightarrow G \text{ cocompact} \\ &\text{s.t. } \iota \in R(\Gamma, G; X) \text{ is not rigid} \\ \Leftrightarrow\; &G \approx \mathrm{SO}(n, 1) \text{ or } \mathrm{SU}(n, 1) \end{aligned}$$

The failure of Rigidity arouses our interest in the deformation space $\mathcal{T}(\Gamma, G; X)$. This corresponds to the Teichmüller theory for Rigidity Theorem A. On the other hand, the above result (Rigidity Theorem B) suggests that such a theory of deformation space may be promising in higher dimension in the **non-Riemannian case**.

On the other hand, in the non-Riemannian case as we explained at the very beginning, we have another difficulty, namely, the action of a 'deformed' discrete subgroup is not always properly discontinuous.

Deformation as discontinuous groups

$\|$

Deformation of discrete subgroups
(group theory)
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In this connection, Goldman [G85] raised a conjecture in the three dimensional Lorentz space form. Namely, he conjectured that there exists a cocompact discontinuous group such that "rigidity" fails but still "stability" holds. Goldman's Conjecture was solved affirmatively by Kobayashi [K98] and Salein [S99].

Goldman's conjecture (see [G85])

$$G = \mathrm{SL}(2,\mathbb{R}), \qquad X = G \times G/\operatorname{diag} G$$
$$(X\colon \text{ Lorentz space form}, \quad \dim X = 3)$$
$$\exists \text{ cocompact discontinuous group } \Gamma \text{ for } X$$
$$\text{s.t. } \begin{cases} \text{Rigidity} & \text{fails,} \\ \text{Stability} & \text{holds.} \end{cases}$$

Solution:Yes (**Kobayashi, Salein**)

The above case treats a non-Riemannian manifold X where the transformation group G is **semisimple**.

Different from the above example, we shall study in this article a deformation of a discontinuous group Γ in the non-Riemannian manifold X where the transformation group G is **nilpotent**. Then, we shall find that there exists a discontinuous group Γ for which both rigidity and stability fails. Such an example in the nilpotent setting can be constructed only if Γ is not a cocompact discontinuous group for X. Loosely, we shall show:

Summary of today's talk (Nilpotent)

$$\exists \ \Gamma \curvearrowright X = \mathbb{R}^{k+1}$$
$$\text{s.t. } \begin{cases} \text{Rigidity} & \text{fails,} \\ \text{Stability} & \text{fails.} \end{cases}$$

More than this, our plan is to explain the explicit structure of $R(\Gamma, G; X)$ and the deformation space $\mathcal{T}(\Gamma, G; X)$ in a certain special setting.

3 Statement of results

Here is a statement of our main results. We take Γ to be $\mathbb{Z}^k$ acting on the Euclidean space $X = \mathbb{R}^{k+1}$ through the following affine transformation group G:

$$G := \left\{ \begin{pmatrix} I_k & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : \begin{array}{c} x,\, y \in \mathbb{R}^k, \\ z \in \mathbb{R} \end{array} \right\} \underset{\substack{\text{2-step} \\ \text{nilpotent} \\ \text{group}}}{\subset} \mathrm{Aff}(\mathbb{R}^{k+1})$$

Here is a brief statement of our main theorem.

<u>**Main Theorem**</u> (see [KN05] for details)

1) (**Failure of Rigidity**)

$$\dim \mathcal{T}(\Gamma, G; X) = \begin{cases} 2k^2 - 1 & (k \equiv 0 \bmod 2), \\ 2k^2 - 2 & (k \equiv 1 \bmod 2,\ k > 1), \\ 2 & (k = 1). \end{cases}$$

2) (**Failure of Stability**)

There is a bijection:

$$R(\Gamma, G; X) \simeq M_1^r \cup M_2^r$$

(M_1^r and M_2^r will be explained later.)

In particular, $\exists\, \varphi$ s.t. $R(\Gamma, G; X)$ is not open near φ.

For the rest of this paper, we will explain some flavor of this theorem and the method of the proof involved.

First, we observe that, a group homomorphism φ is determined by its value at the generators. Let $\{e_1, \cdots, e_k\}$ be a standard basis of Γ. Then, by looking at the values at e_j $(1 \leq j \leq k)$, we may regard $\mathrm{Hom}(\Gamma, G)$ as a subset of the direct product of k copies of G.

Then, we ask how $R(\Gamma, G; X)$ can be characterized as a subset of this direct product:

$$\begin{array}{ccc} \varphi & \longmapsto & (\varphi(e_1), \ldots, \varphi(e_k)) \\ \pitchfork & & \pitchfork \\ \mathrm{Hom}(\Gamma, G) & \hookrightarrow & \overbrace{G \times \cdots \times G}^{k} \\ \cup & & \cup \\ R(\Gamma, G; X) & \simeq & \boxed{\quad ? \quad} \end{array}$$

Characterize!

We recall that G is a $2k+1$ dimensional nilpotent Lie group. We shall use the following coordinates of G:

$$\begin{array}{cccc} g: & \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R} & \longrightarrow & G \\ & \cup & & \cup \\ & (\vec{x}, \vec{y}, z) & \mapsto & \exp \begin{pmatrix} 0_k & \vec{x} & \vec{y} \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

We define the following sets as:

$$M_1^r := \{(\vec{x}, Y, \vec{z}) \in M(k, k+2; \mathbb{R}) : \vec{z} \neq \vec{0},\ \mathrm{rank} \begin{pmatrix} Y \\ {}^t\vec{z} \end{pmatrix} = k\}$$

$$M_2^\tau := \{(X,Y) \in M(k,2k;\mathbb{R}) : \det(Y-\lambda X) \neq 0 \text{ for } \forall\, \lambda \in \mathbb{R}\}.$$

We define:

$\Psi_1 : M_1^\tau \hookrightarrow G \times \cdots \times G$
by
$\Psi_1(\vec{x}; Y; \vec{z})_j := g(z_j\vec{x}, \vec{y_j}, z_j) \quad (1 \leq j \leq k)$

$\Psi_2 : M_2^\tau \hookrightarrow G \times \cdots \times G$
by
$\Psi_2(X,Y)_j := g(\vec{x_j}, \vec{y_j}, 0) \quad (1 \leq j \leq k)$

Here, $X = (\vec{x_1}, \cdots, \vec{x_k})$, $Y = (\vec{y_1}, \cdots, \vec{y_k})$, ${}^t\vec{z} = (z_1, \cdots, z_k)$

Then, the second statement of our main theorem is stated in a more precise way. That is, the parameter space $R(\Gamma, G; X)$ is exactly the disjoint union of M_1^τ and M_2^τ through the maps Ψ_1 and Ψ_2:

Description of $R(\Gamma, G; X)$

$$R(\Gamma, G; X) \underset{\Psi_1 \cup \Psi_2}{\xleftarrow{\sim}} M_1^\tau \cup M_2^\tau$$

4 Idea of proof

We mention briefly our idea of the proof. See [KN05] for details.

Step 1. (Description of $R(\Gamma, G; X)$)
The most important part is the description of the parameter set of discontinuous groups $R(\Gamma, G; X)$. This consists of two subproblems. The first one is the deformation of discrete subgroups, and the second thing is to tell which discrete subgroup acts properly discontinuously and which one does not act properly discontinuously.

The latter problem can be solved by using the criterion of properly discontinuous action. For this we can use Lipsman's conjecture for 2-step nilpotent Lie group, which is now a theorem [N01]. We also note that Lipsman's conjecture is also true for 3-step nilpotent cases. This was recently proved by Yoshino [Y05] and Baklouti-Fatma [BK05] independently .

Here, we recall a characterization of proper actions on nilpotent homogeneous spaces:

Lipsman's Conjecture ([L95]). Let G be a simply connected nilpotent Lie group, and H, L be its connected subgroups. Then the following holds.

$L \curvearrowright G/H$ is proper
$\Leftrightarrow \; L \cap gHg^{-1} = \{e\}$ for $\forall\, g \in G$

Here is some remark about how to apply the solutions to Lipsman's Conjecture. For this, we need to compare discrete groups with connected groups. In fact, Lipsman's Conjecture deals with the actions of connected groups. But our interest is the action of discrete groups. A simple way to bridge them is to find an appropriate Lie group $\overline{\Gamma}$ that contains a discrete subgroup. Then we may expect that if Γ acts properly discontinuously then $\overline{\Gamma}$ acts properly. Unfortunately, such a statement fails if G is semisimple. However, fortunately, this is true in our setting. Namely, we can use the following lemma in our case when $X = \mathbb{R}^{k+1}$ is regarded as a homogeneous space of the Lie group G:

$$\boxed{\begin{array}{c} \Gamma \subset {}^{\exists}L\text{: connected subgroup} \\ \text{s.t. } L \supset \Gamma \text{ cocompact} \end{array}}$$

Then, Γ acts properly discontinuously on X if and only if L acts properly on X (see [K89]). Thus, we can concentrate on the deformation of a connected subgroup L under the assumption that L acts properly on X. This assumption can be verified by applying Lipsman's Conjecture ('theorem' in this setting).

Step 2. (Stability fails)
This step can be proved by an explicit description of $R(\Gamma, G; X)$ done in Step 1.

Step 3. (Description of Deformation space)
The description of deformation space can be carried out by finding how G acts on our parameter spaces M_1^r and M_2^r.

Step 4. (Dimension formula of $\mathcal{T}(\Gamma, G; X)$)
The final step is an easy consequence of Step 3 and linear algebra.

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